Readers' Forum

Comment on "Improved Series Solutions of Falkner-Skan Equation"

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IN Ref. 1, Afzal presents solutions of the Falkner-Skan equation

$$f''' + ff'' + \beta(' - f'^2) = 0 \tag{1a}$$

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1 \tag{1b}$$

in terms of expansions in powers of β. It is the purpose of this Comment to show that this is a particulary difficult and awkward approach to a mathematically trivial problem. Elements of the exploitation of the contraction mapping properties of the Falkner-Skan equation viewed as an operator on f'' were first noted by Weyl[2-4] and fully exploited in the solution of self-similar, binary gas, compressible, laminar boundary layers by Wortman.[5] In the present case, when an integrating factor

$$E = \exp\left(\int_0^\eta f \mathrm{d}\eta\right) \tag{2}$$

is used, formal integration of Eq. (1a) yields

$$f'' = E^{-1}\left[K - \beta\int_0^\eta (1 - f'^2)E\mathrm{d}\eta\right] \tag{3}$$

with K being the constant of integration determined from the integration of Eq. (3) between the limits of 0 and ∞. Thus,

$$K = \left[1 - \beta\int_0^\infty E^{-1}\int_0^\eta (1 - f'^2)\mathrm{d}x\mathrm{d}\eta\right] \Big/ \int_0^\infty E^{-1}\mathrm{d}\eta \tag{4}$$

and f' and f are obtained from successive quadratures. The iteration sequence may be started from an arbitrary f'' and Simpson's rule or trapezoidal integration is sufficient for five-place accuracy in $f''(0)$. Equation (3) is, in fact, a contraction map for moderate values of β and the iteration sequence is stabilized through weighted averaging up to $\beta = 20$.[6] Three-dimensional foreign gas injection applications and three-dimensional turbulent flows were exhibited in Refs. 7 and 8, respectively. In all cases, extremely simple compact computer codes were used and convergence was achieved in about 1 s of IBM 370 computer time. The basic Falkner-Skan equation can be programmed in about 24 lines of code.

The efforts of Ref. 1 represent exemplary industriousness worthy of admiration, but the contribution to a problem-solving-oriented profession such as engineering is not obvious, and thus the place of the Note in the *AIAA Journal* must be questioned.

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Reply by Author to A. Wortman

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IN his Comment, Wortman has attempted a comparison of the time and effort involved in improving the convergence of the series solution[1] of the Falkner-Skan equation due to Aziz and Na[2] with purely numerical solutions. The technique of computer-extended series[1,2] is relatively recent. In a recent review on the subject, Van Dyke[3] has concluded that "clearly the technique of computer-extended series is not yet ready to be applied routinely. Until we understand it better it should be regarded as a useful but dangerous tool, to be used with utmost care and skepticism."

For the Falkner-Skan equation, the eleven-term series solution in powers of β was improved[2] for $-0.19884 \leq \beta \leq 2$ by the application of the Shanks transformation, which does not exploit the analytical structure of the series. If the series permits an estimate of the analytical structure, there are more effective ways of improving the series.[4] Therefore, it was logical to explore the analytical structure of Aziz and Na's[2] series (location and nature of the nearest singularity), which formed a natural basis for the improvement of its convergence, once and for all values of β. Recasting of the original series[2] by Euler transformation or by subtraction of the singularity carried out by Afzal[1] is not only fairly simple and straightforward but also predicts very good results for all values of β in the range $-0.19884 \leq \beta \leq \infty$.

Regarding the so-called mathematically trivial Falkner-Skan equation, the work of Cebeci and Keller[5] (where the

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boundary layer was divided into three intervals and estimation of seven values was necessary instead of one in order to reduce the sensitivity of the solution to an initial guess) and the quasilinearization approach of Zagustin et al.[6] are of some concern (see also Hartman[7]). In fact, the Falkner-Skan equation is a contraction map[8] for moderate positive values of β. Wortman[9,11] and Wortman and Mills[10] have studied the flow with favorable pressure gradients $\beta>0$ and employed weighted averaging[9,10] in order to stabilize the iteration sequence up to $\beta=20$. Further, for $\beta<0$, the Falkner-Skan equation has certain special features described below that are absent for $\beta>0$.

1) For $\beta<0$, the decay of the solutions at infinity is mixed: exponential as well as algebraic. It is well known that for given $\beta<0$ there is a family of solutions each of which satisfies the boundary conditions. To select the appropriate solution, i.e., one with exponential decay at infinity, one has to look beyond the equations.[12]

2) For $-0.19884\leq\beta<0$, the solutions are dual (with a turning point at $\beta=-0.19884$) corresponding to forward and reverse flows in the boundary layer.[12] Special care is needed in the development of an algorithm for computing the dual solutions.[13]

3) For $\beta<-0.19884$, the solution has an infinite number of discontinuous branches[14] where velocity overshoot occurs. It is highly unlikely that the type of code in question can handle these issues in a satisfactory manner.

The work,[1] apart from its own intrinsic interest in the technique of computer-extended series also predicts the square-root singularity anticipated by Stewartson,[12] determines the amplitude of the singularity for the first time, and the duality of solutions for $\beta<0$. In addition, it also gives the closed-form solutions for skin friction, which are very accurate even for very large values of β.

In conclusion, the Comment arose from a lack of appreciation of the character of the Falkner-Skan equation for negative values of β and the technique of computer-extended series.

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